Gravitational waves in an anomaly-induced inflation

J.C. Fabris [a], A.M. Pelinson, I.L. Shapiro and F.I. Takakura [b]

[a]Universidade Federal do Espírito Santo, Departamento de Física, Espírito Santo, Brazil

[b]Departamento de Fisica, ICE, Universidade Federal de Juiz de Fora, MG, Brazil

The behaviour of gravitational waves in the anomaly-induced inflationary phase is studied. The metric perturbations exhibit a stable behaviour, with a very moderate growth in the amplitude of the waves. The spectral indice is computed, revealing an almost flat spectrum.

The goal of the present study is to analyze the fate of gravitational waves in a background defined by an inflationary scenario created by a trace anomaly induced by quantum effects generated by matter fields in the primordial Universe [1]. This model is a generalization of the Starobinsky model [2], where a similar mechanism has been developed. The counterterms necessary to avoid divergences in the quantization of these conformal quantum fields in a Friedmann-Robertson-Walker background leads to a higher derivative action with a trace anomaly. Details of the model may be found in [1]. An important point concerns the fact that an inflationary phase may be obtained with this effective gravitational action. When all fields are massless, this inflationary phase is an eternal de Sitter phase, where many of the traditional problems of the inflationary scenario appear. However, if massive conformal fields are included, the inflationary phase has more complicated form. A detailed analysis of the massive case reveals that a huge amplification of the scale factor may be obtained, of the order of 10^{10} e-folds. However, from the observational point of view, only the last $65-70$ e-folds are relevant. During this final period, the behaviour of the scale factor may be approximated by an exponential function, the Hubble parameter being constant. It is possible that a transition to the FRW standard scenario may be achieved solving the graceful exit problem [4].

The equation governing the behaviour of gravitational waves in the anomaly-induced model, in terms of the cosmic time t, is [3]

$$b_0 \ddddot{h} + b_1 \dddot{h} + b_2 \ddot{h} + b_3 \dot{h} + b_4 h$$

$$+ n_1 \frac{\nabla^2 \dot{h}}{a^2} + n_2 \frac{\nabla^2 \ddot{h}}{a^2} + n_3 \frac{\nabla^4 h}{a^4} \;=\; 0\,, \qquad (1)$$

with

$$b_0 = \tilde{b}_0\,, \quad b_1 = H\tilde{b}_1\,, \quad b_2 = H^2\tilde{b}_2\,,$$

$$b_3 = H^3\tilde{b}_3\,, \quad b_4 = H^4\tilde{b}_4\,,$$

$$n_1 = H\tilde{n}_1\,, \quad n_2 = \tilde{n}_2\,, \quad n_3 = \tilde{n}_3\,. \qquad (2)$$

Here the quantities with tildes are pure numbers:

$$\tilde{b}_0 = 1\,, \quad \tilde{b}_1 = 6\,, \quad \tilde{b}_2 = 11\,, \quad \tilde{b}_3 = 6\,,$$
$$\tilde{n}_1 = -2\,, \quad \tilde{n}_2 = -2\,, \quad \tilde{n}_3 = 1\,, \qquad (3)$$

while $\tilde{b}_4$ depends on the multiplet content of the matter fields:

$$b_4 = \frac{12}{(4\pi)^2}\left(\frac{N_0}{360} + \frac{11N_{1/2}}{360} + \frac{31N_1}{180}\right), \qquad (4)$$

where N_0, $N_{1/2}$ and N_1 are the scalar, fermionic and vectorial numbers of matter fields. For example, in the minimal standard model, $N_0 = 4$, $N_{1/2} = 24$ and $N_1 = 12$, leading to $b_4 \sim 0.2$.

As usual, $H = \dot{a}/a$ is the Hubble parameter. The parameters described above take those value in the last 65 e-fold. In this case, the background model enters in a quasi-de Sitter and the scale factor behaves essentially as

$$a \sim e^{Ht} \qquad (5)$$

where, in an appropriate GUT model,

$$H \approx 10^{-5} M_{Pl} \, .$$

Changing to the conformal time η such that $dt = a d\eta$, the scale factor takes the form

$$a \sim -\frac{1}{H_0 \eta} \tag{6}$$

with $\eta < 0$. The Universe expands as $\eta \to 0_-$. In terms of this new time parameter, and performing a Fourier decomposition of the function $h = h(x,\eta) = h(\eta)e^{i\vec{k}.\vec{x}}$, k being the wavenumber of the perturbation, we obtain the following fourth order differential equation describing the evolution of gravitational waves.

$$h^{iv} + 2k^2 h'' + \biggl\{\frac{\tilde{b}_4}{\eta^4} + k^4\biggr\} h = 0 \, . \tag{7}$$

The units are such that $k = 1$ implies a comoving wavelength of the perturbation of the order of the Planck's length. Let us notice the remarkably simple form of the equation (7), which is essentially simpler that the equivalent equation in terms of the physical time (1).

We first investigate the behaviour of h in the two asymptotic regimes: small and large wavenumbers. For large wavenumbers $k >> 1$ and we may approximate the equation to

$$h^{iv} + 2k^2 h'' + k^4 h = 0 \, , \tag{8}$$

whose solution is

$$h = c_\pm e^{\pm ik\eta} + c'_\pm \eta e^{\pm ik\eta} \, . \tag{9}$$

Since η approaches zero as the universe expands, the solution above is a combination of oscillatory and decreasing oscillatory modes. A quantum state can be implemented in the initial state (when $\eta \to -\infty$), if, for example, $c_- = c_0/\sqrt{2k}$, $c_0 =$ constant, $c_+ = c'_\pm = 0$. For small wavenumbers, on the other hand, the equation simplifies to

$$h^{iv} + \frac{\tilde{b}_4}{\eta^4} h = 0 \, . \tag{10}$$

We can look for solutions under the form $h \propto \eta^p$, where p is a number. In this case, p obeys the algebraic relation

$$p(p-1)(p-2)(p-3) + \tilde{b}_4 = 0 \, . \tag{11}$$

There is no root for p with $Re\, p < 0$. Hence, there is only decreasing modes in the long wavelength approximation.

All the behaviours sketched above were confirmed through numerical integration. The results indicate that the inflationary phase in this model is quite stable with respect to tensorial perturbations. There is a small production of gravitons during the inflationary phase. This seems to be a positive result since a very large production of graviton during the inflationary phase would lead to undesirable consequences since a large amplification of gravitational waves brings the problem of back reaction [5] and, at same time, renders the linear approximation doubtful. No such problems exist in the behaviour of gravitational waves in the context of the anomaly-induced inflationary scenario.

It is instructive to compare the model described above with the results obtained with the case of the standard inflation based on a cosmological constant term (or an evolving scalar field) [6]. Since a de Sitter phase is a good approximation to the standard case, the scale factor behaves as before. In this case the equation governing the behaviour of gravitational waves takes the form

$$h'' - 2\frac{a'}{a}h' + \biggl\{k^2 - 2\biggl[\frac{a''}{a} - \biggl(\frac{a'}{a}\biggr)^2\biggr]\biggr\} h = 0 \, . \tag{12}$$

The solution for this equation is

$$h = \frac{1}{\sqrt{\eta}}\biggl\{c_1 H^{(1)}_{3/2}(k\eta) + c_2 H^{(2)}_{3/2}(k\eta)\biggr\} \, , \tag{13}$$

where the $H^{(1,2)}_\nu$ are Hankel functions of first and second kind, and the c's are constants.

From the point of view of gravity production, directly connected with the amplification of the gravitational waves, the anomaly-induced inflationary model is very different from the standard one. While the solutions (13) exhibit an amplification of the order 10^{65} in the last 65 e-folds, in the anomaly-induced inflation the amplitude of the gravitational waves remains essentially stable. The huge amplification of gravitational waves in the standard inflationary scenario brings essentially two problems: first, the non-linear regime is reached very quickly unless the initial pertur-

bations are excessively small; moreover, it is difficult to ignore in this situation the back-reaction. The firsts problem may be coped with through the quantum normalization: the initial amplitude must be of the order of the inverse Planck's mass. The second one is object of many investigations today [5]. These problems are absent in the anomaly-induced inflationary model: since a perturbation originated in the beginning of the de Sitter phase would have been amplified at the end of this phase by a factor of order of unity, the perturbation enters in the radiative phase very small. During the radiative phase, the equation governing the evolution of gravitational waves is given by (12), with $a \propto \eta$ and with $a \propto \eta^2$ during the matter dominated phase. The gravitational waves will be considerably amplified only during these phases. The non linear regime may be attained only quite recently.

We turn now to the problem of evaluation of the spectral indice of the power spectrum in the anomaly-induced inflationary scenario. The power spectrum is defined as

$$P^2(k) = k^3\delta_k^2 \,. \qquad (14)$$

The spectral indice is obtained through the relation

$$n_T = \frac{d\ln P_k^2}{d\ln k}\,, \qquad (15)$$

where the subscript T indicates that we are evaluating the power spectrum for the tensorial modes. A scale invariant spectrum is characterized by the fact that the amplitude of the perturbation are the same for any value of k when the perturbation cross the horizon. This implies $n_T = 0$, when the expression (14) is evaluated at the horizon crossing.

A fundamental point in the analysis to be performed is the question of initial conditions. In the anomaly-induced inflation, as in the standard inflaton-based inflation, the seeds of the initial perturbations are quantum mechanical. The normalized initial quantum spectrum takes the form

$$h(\eta)_k \sim \frac{e^{-ik\eta}}{\sqrt{2k}} \quad . \qquad (16)$$

Let us consider the initial time $\eta_i \sim 1$ in Planck's unities. The final time is $\eta_f \sim 10^{-30}$. Suppose also that the inflationary phase ends at about $t \sim 10^{-38}\,s$, consistent with the fact that $H_0 \sim 10^{-5}M_{Pl}$. Since in our unities $k \sim 1$ means a initial perturbation of the order of the Planck's scale, perturbations today in range from the Hubble horizon and some hundreds of megaparsecs, evaluated today, implies initial perturbations with $2\pi 10^{-3} < k < 2\pi 10^{-1}$. We now compute numerically the perturbations using the initial conditions (16) for the traditional inflationary scenario and for the anomaly-induced inflation. In both cases, the spectrum is essentially flat: $n_T \sim 0$ at the end of the inflationary phase. Considering, for example, $\pi/100 < k < \pi/10$, the spectral indice is $n_T \sim -0.027$. This may be compared with the spectral indice for the standard inflationary scenario which, for a de Sitter phase, is strictly zero in the large wavenumber approximation.

REFERENCES

1. J.C. Fabris, A.M. Pelinson and I.L. Shapiro, Nucl. Phys. **B597** (2001) 539; see also presentations by I. Shapiro and A. Pelinson et al in this volume.
2. A.A. Starobinski, JETP Lett. **30** (1979) 719.
3. A.M. Pelinson, I.L. Shapiro and F.I. Takakura, Nucl. Phys. **B648** (2003) 417.
4. I.L. Shapiro, Phys. Lett. **B530** (2002) 10.
5. R.H. Brandenberg, *Back reaction of cosmological perturbations*, hep-th/0004016.
6. E.W. Kolb and M. Turner, *The very early universe* (Addison-Wesley, New York, 1994).